Theoretical Foundations of General Stochastic Hybrid Processes
Work Package SHS, Deliverable DSHS2

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Chapter 1 Introduction

Task SHS2: Lay the foundations for reachability analysis of PDMP. Fundamental problems that will be addressed here include the development of appropriate measures on the space of trajectories of these systems to capture reachability questions and conceptual algorithms for computing the measure of reachability ”events”.

Part of this task was already accomplished in DSHS1 where fundamental questions relating to the measurability event for PDMP were addressed. In DSHS2 we extend these approach to a very general class of stochastic hybrid processes that includes as a special case all the classes identified in DSHS1: Piecewise Deterministic Markov Processes (PDMP, [12]), Stochastic Hybrid Systems (SHS, [16]) and Switching Diffusion Processes (SDP, [15]).

We formulate very general class of Markov processes, which will be called *Markov Strings*, loosely based on the so-called “melange” of Markov processes introduced by Meyer [21]. We start with a countable family of Markov processes which have nice properties (the strong Markov property, the càdlàg property). For each process belonging to this family, we underlie the associated probabilistic elements: probability space, natural filtration, translation operator, probabilities on the trajectories. The probabilistic construction of the Markov String is quite natural:

1. start with one process which belongs to the given family;
2. kill the starting process at the time T_1 of the first jump;
3. jump according to a renewal kernel which gives the distribution of the post-jump location;
4. “restart” an other process (belonging to the given family) corresponding to the new location;
5. proceed until the first jump of the new process and kill again, etc.

The pieced together process obtained by the above procedure is called Markov String. Its jump structure is completely described by a renewal kernel which is priory given and a family of stopping times associated with the initial processes. We suppose that the resulted Markov String has finitely many jumps in finite time. We prove that the above Markov Strings, as stochastic processes, enjoy useful properties like the (strong) Markov property and the càdlàg property. The class of processes considered here differ from the class considered in [21] in that:

- The jump times are essentially given stopping times, *not necessarily the life times of the component processes*;
- After a jump the string restarts following an other process which might be different from the pre-jump process;
- The mixing (“melange”) operation in [21] is only sketched and the author claims that it can be obtained using the renewal (“renaissance”) operation. We consider that the passing from renewal to mixing is not straightforward. It is necessary to emphases the construction of all probabilistic elements associated with the resulted string. Lifting the renewal construction to the mixing construction we found that remarkable changes should be introduced in the definition of the state space, probability space, probability measures on the trajectories corresponding to the resulted Markov string.

We then show how a very general class of stochastic hybrid processes can be embedded in the framework of Markov Strings. This class (which was referred to as the General Stochastic Hybrid Model (GSHM) in DSHS1) allows:

1. Diffusion processes in the continuous evolution.
2. Spontaneous discrete transitions (according to a transition rate).
3. Forced transitions (driven by a boundary hitting time).
4. Probabilistic reset of the discrete and continuous state as a result of discrete transitions.

As discussed in DSHS1 the class of GSHM includes as special cases all the classes of stochastic hybrid processes mentioned above (PDMP, SHS and SDP). We show that the class of GSHM inherits the strong Markov and càdlàg properties from Markov Strings. We also develop the infinitesimal generator for this class of stochastic hybrid models.

Applying the results of DSHS1, the above properties imply the universal measurability of the reach event for Borel measurable subsets of the state space. Extending the approach outlined in DSHS1 further, we discuss how bounds on the measure of the reach event can be computed using the generator and the corresponding Dirichlet forms. It has already been proved in the literature that Dirichlet forms [14, 18] constitute a powerful tool for studying Markov processes (see, for example, [1, 18] and the references therein). Dirichlet form techniques have found striking applications in the study of stochastic partial differential equations [1, 7]. This is mainly due to the fact that they allow to develop a highly nontrivial stochastic analysis under some minimal regularity hypothesis, for instance, on very irregular spaces without differentiable structure like fractals, or on infinite dimensional spaces like path spaces or spaces of measures.

For Dirichlet forms, a lot of work was carried out on axiomatizations and representation results. This provides a mathematical vehicle for zooming in and out at different levels of abstraction in a consistent way. For example, in the most abstract view, the Dirichlet spaces can be seen as mixing a linear space structure with a partial order structure, by providing simple compatibility axioms. In more concrete applications a Dirichlet space defines a logical type of functions with an inner product given explicitly by a logical expression. The advantage of Dirichlet forms which derives from this is that they can be easily implemented. There are two main streams: one is symbolic (like using a model-checker or a theorem prover or their combination like PVS [13]) and another one is numerical [10]. For the reachability problem the symbolic approach has been intensively applied (see e.g. the papers in [19]), especially because the accessible states can be generated. In the case of PVS, we can link these techniques with the huge mathematical libraries made available by the theorem provers.

The material is arranged in Chapters. Chapter 2 presents the theoretical foundations of Markov strings. The main properties of these stochastic processes are given with the full proofs. Chapter 3 presents GSHM, as a class of stochastic processes which generalize the models introduced by Davis in [12]. The difference between GSHM and PDMP is that for GSHM between two consecutive jumps the process is a diffusion whilst for PDMP the inter-jumps motion is deterministic, according to a vector field. In chapter 4, we employ the induced Dirichlet forms to obtain bounds on the measure of the reach event.

Chapter 2 Markov Strings

In this chapter we define the Markov string notion, which, roughly speaking, is a stochastic process obtained by mixing some given Markov processes.

We prove that if we start with a countable family of 'nice' Markov processes then the Markov string resulted will inherit the properties of its components.

2.1 The Ingredients

Suppose that $\mathbb{M}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, x^i_t, \theta^i_t, P^i, P^i_{x^i})$, $i \in Q$ is a countable family of Markov processes. We denote the state space of each $\mathbb{M}^i$ by $(X^i, \mathcal{B}^i)$ and assume that $\mathcal{B}^i$ is the Borel σ-algebra of X^i if X^i is a topological Hausdorff space. Let Δ be the cemetery point for all X^i, $i \in Q$, which is an adjoined point to X^i, $X^i_\Delta = X^i \cup \{\Delta\}$. The existence of Δ is assumed in order to have a probabilistic interpretation of $P^i_{x^i}(x^i_t \in X^i) < 1$, i.e. at some 'termination time' $\zeta^i(\omega_i)$ the process $\mathbb{M}^i$ escapes to and is trapped at Δ. For each $i \in Q$, the elements $\mathcal{F}^i$, $\mathcal{F}^{i,0}_t$, $\mathcal{F}^i_t$, $\theta^i_t, P^i, P^i_{x^i}$ have the usual meaning as in the Markov process theory [6]:

- $(\Omega^i, \mathcal{F}^i, P^i)$ denotes the underlying probability space
- $\mathcal{F}^{i,0}_t$ denote the *natural filtration*, i.e. $\mathcal{F}^{i,0}_t = \sigma\{x^i_t, s \le t\}$ and $\mathcal{F}^{i,0}_\infty = \vee_t \mathcal{F}^i_t$.
- $x^i_t : (\Omega^i, \mathcal{F}^i) \to (X^i, \mathcal{B}^i)$ is a $\mathcal{F}^i/\mathcal{B}^i$-measurable function for all $t \ge 0$.
- $\theta^i_t : \Omega^i \to \Omega^i$, for all $t \ge 0$, is the *translation operator*, i.e.

$$x^i_s \circ \theta^i_t = x^i_t,\ t, s \ge 0$$

- $P^i_{x^i} : (\Omega^i, \mathcal{F}^i) \to [0,1]$ is a probability measure (so-called *Wiener probability*) such that $P^i_{x^i}(x^i_t \in E^i)$ is $\mathcal{B}^i$-measurable in $x^i \in X^i$ for each $t \ge 0$ and $E^i \in \mathcal{B}^i$.
- If $\mu^i \in \mathcal{P}(X^i_\Delta)$, i.e. μ^i is a probability measure on $(X^i, \mathcal{B}^i)$ then we can define

$$P^i_{\mu^i}(\Lambda) = \int_{X^i_\Delta} P^i_{x^i}(\Lambda)\mu^i(dx^i), \Lambda \in \mathcal{F}^{i,0}.$$

  We then denote by $\mathcal{F}^i_\infty$ (resp. $\mathcal{F}^i_t$) the completion of $\mathcal{F}^{i,0}_\infty$ (resp. $\mathcal{F}^{i,0}_t$) with respect to all $P^i_{\mu^i}$, $\mu^i \in \mathcal{P}(X^i_\Delta)$.
- We say that a family $\{\mathcal{M}^i_t\}$ of sub-σ-algebras of $\mathcal{F}^i$ is an admissible filtration if $\mathcal{M}^i_t$ is increasing in t and $x^i_t \in \mathcal{M}^i_t/\mathcal{B}^i$ for each $t \ge 0$. Then $\mathcal{F}^{i,0}_t$ is the *minimum admissible filtration*. An admissible filtration $\{\mathcal{M}^i_t\}$ is right continuous if $\mathcal{M}^i_t = \mathcal{M}^i_{t+} = \cap\{\mathcal{M}^i_{t'} | t' > t\}$.
- Given an admissible filtration $\{\mathcal{M}^i_t\}$, a $[0,\infty]$-valued function τ^i on Ω^i is called an $\{\mathcal{M}^i_t\}$-*stopping time* if $\{\tau \le t\} \in \mathcal{M}^i_t$, $\forall t \ge 0$.
- For an admissible filtration $\{\mathcal{M}^i_t\}$, we say that $\mathbb{M}^i$ is *strong Markov* with respect to $\{\mathcal{M}^i_t\}$ if $\{\mathcal{M}^i_t\}$ is right continuous and

$$P^i_{\mu^i}(x^i_{\tau+t} \in E^i | \mathcal{M}_\tau) = P_{x^i_\tau}(x^i_t \in E^i);\ P^i_{\mu^i} - a.s.$$

  $\mu^i \in \mathcal{P}(X^i_\Delta)$, $E^i \in \mathcal{B}^i$, $t \ge 0$, for any $\{\mathcal{M}^i_t\}$-stopping time.
- $\mathbb{M}^i$ has the *càdlàg property* if for each $\omega_i \in \Omega^i$, the sample path $t \mapsto x^i_t(\omega_i)$ is right continuous on $[0,\infty)$ and has left limits on $(0,\infty)$ (inside X^i_Δ).
- Let (P^i_t) denote the operator semigroup associated to $\mathbb{M}^i$ which maps $\mathcal{B}^i(X^i)$ into itself given by

$$P^i_t f(x^i) = E^i_{x^i} f(x^i_t),$$

  where $E^i_{x^i}$ is the expectation with respect to $P^i_{x^i}$. Then a function f is *p-excessive* if it is non-negative and $e^{-pt}P^i_t f \le f$ for all $t \ge 0$ and $e^{-pt}P^i_t f \nearrow f$ as $t \searrow 0$.

Assumption 1 *Suppose that, for each $i \in Q$,*

1. $\mathbb{M}^i$ is a strong Markov process.

2. P^i is a complete probability.

3. The state space X^i is a topological space which is homeomorphic to a Borel subset of a complete separable metric space (Borel space).

4. $\mathbb{M}^i$ enjoys the càdlàg property.

5. The p-excessive functions ($p > 0$) of the process are almost surely right continuous on the trajectories.

Part (3) implies that the underlying probability space Ω^i can be assumed to be $D_{[0,\infty)}(X^i)$, the space of functions mapping $[0, \infty)$ to X^i which are right continuous functions with left limits. In the terminology of [20], parts (1), (3) and (5) of the Assumption 1 imply that each $\mathbb{M}^i$ is a *Borel right process*.

Using this family of Markov processes $\{\mathbb{M}^i\}_{i\in Q}$, we define a new Markov process whose realizations consist of concatenations of realizations for different $\mathbb{M}^i$. To achieve this goal, we need to define the transition mechanism from one process to the others. The switching mechanism will be given by:

1. a family of stopping times, a stopping time for each process (which gives the switching temporal parameter),

2. a renewal kernel which gives the post jump location.

2.2 The Construction

Using the elements defined in the section 2.1 we construct a stochastic process $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$. The process $\mathbb{M}$ is obtained by the concatenation of the component processes $\mathbb{M}^i$ and will be called *Markov String*. Roughly speaking, this Markov string is constructed in such a way that its sample paths are obtained sticking the sample component paths between some switching times. The switching times can be obtained using a given sequence of stopping times associated to the component processes.

To completely define the Markov String we need to specify the following elements

1. $(X, \mathcal{B})$ - the state space;

2. $(\Omega, \mathcal{F}, P)$ - the underlying probability space;

3. $\mathcal{F}_t$ - the natural filtration;

4. θ_t - the translation operator.

5. P_x - Wiener probabilities.

2.2.1 State Space $(X, \mathcal{B})$

The state space will be X defined as follows. X is constructed as the direct sum of spaces X^i, with the same cemetery point Δ, i.e.

$$X = \bigcup_{i\in Q} \{(i, x) | x \in X^i\}. \tag{2.1}$$

It is possible to define a metric ρ on X such that $\rho(x_n, x) \to 0$ as $n \to \infty$ with $x_n = (i_n, x_n^{i_n})$, $x = (i, x^i)$ if and only if there exists m such that $i_n = i$ for all $n \geq m$ and $x^i_{m+k} \to x^i$ as $k \to \infty$. The metric ρ restricted to any component X^i is equivalent to the usual component metric [12]. Each $\{i\} \times X^i$, being a Borel space, will be homeomorphic to a measurable subset of the Hilbert cube, $\mathcal{H}$ (Urysohn's theorem, Prop. 7.2 [5]). Recall that $\mathcal{H}$ is the product of countable many copies of $[0, 1]$. The relation (2.1) implies that X will be, as well, homeomorphic to a measurable subset of $\mathcal{H}$. Thus X is a Borel space [5].

The space X can be endowed with the Borel σ-algebra $\mathcal{B}(X)$ generated by its metric topology. Moreover, we have

$$\mathcal{B}(X) = \sigma\{\bigcup_{i\in Q} \{i\} \times \mathcal{B}^i\}. \tag{2.2}$$

Then $(X, \mathcal{B}(X))$ is a Borel space, whose Borel σ-algebra $\mathcal{B}(X)$ restricted to each component X^i gives the initial σ-algebra $\mathcal{B}^i$ [12]. The above argument allows us to make the following remark:

Remark 1 *We can suppose, without loss of generality, that $X^i \cap X^j = \emptyset$ if $i \neq j$. Thus the relations (2.1) and (2.2) become*

$$X = \bigcup_{i \in Q} X^i; \tag{2.3}$$

$$\mathcal{B}(X) = \sigma(\bigcup_{i \in Q} \mathcal{B}^i). \tag{2.4}$$

Therefore, we can suppose, as well, that $\Omega^i \cap \Omega^j = \emptyset$ if $i \neq j$.

2.2.2 Probability Space

The space Ω can be thought as the space generated by the concatenation operation defined on the union of the spaces Ω^i (which pairwise disjoint), i.e.

$$\Omega = \left(\bigcup_{i \in Q} \Omega^i \right)^* .$$

In a general setting, Ω can be identified with the product $\prod_{i \in Q} \Omega^i$. Thus, the $\sigma-$algebra $\mathcal{F}$ on Ω will be the $\sigma-$algebra product of $\mathcal{F}^i$, i.e. $\mathcal{F} = \prod_{i \in Q} \mathcal{F}^i$. The probability P on $\mathcal{F}$ will be defined as a product measure, i.e. $P = \prod_{i \in Q} P^i$.

Let $\widehat{\mathcal{F}}$ be the $\sigma(\bigcup_{i \in Q} \mathcal{F}^i)$ defined on $\bigcup_{i \in Q} \Omega^i$.

2.2.3 Recipe

Formally, in order to define the desired Markov string, $\mathbb{M}$, we need to give:

1. $(S^i)_{i \in Q}$, where, for each $i \in Q$, S^i is a stopping time of $\mathbb{M}^i$ with the 'memoryless' property, i.e.

$$S^i(\theta^i_t \omega_i) = S^i(\omega_i) - t, \forall t < S^i(\omega_i) \tag{2.5}$$

2. The switching mechanism between the processes $\mathbb{M}^i$ is governed by a *renewal kernel* which is a Markovian kernel

$$\Psi : \{\bigcup_{i \in Q} \Omega^i\} \times \mathcal{B}(X) \to [0, 1]$$

   satisfying the following conditions:

   (a) If $S^i(\omega_i) = +\infty$ then $\Psi(\omega_i, \cdot) = \varepsilon_\Delta$;

   (b) If $t < S^i(\omega_i)$ then $\Psi(\theta^i_t \omega_i, \cdot) = \Psi(\omega_i, \cdot)$.

Notation. The cemetery point of the state space Ω^i is denoted by $[\Delta]^i$ and the cemetery point of Ω is denoted by $[\Delta]$. We use to denote by ω (resp. $\widehat{\omega}$ or ω_i) an arbitrary element of Ω (resp. $\bigcup_{i \in Q} \Omega^i$ or Ω^i).

In the following, we give the procedure to construct a sample path of the stochastic process $(x_t)_{t>0}$ with values in X, starting from a fixed initial point $x_0 = x_0^{i_0} \in X^{i_0}$. Let ω_{i_0} be a sample path of the process $(x_t^{i_0})$ starting with x_0. In fact, we give a recipe *to construct a Markov string starting with an initial path* ω_{i_0}. Let $T_1(\omega) = T_1(\omega_{i_0}) = S^{i_0}(\omega_{i_0})$. The sample path $x_t(\omega)$ up to the first jump time is now defined as follows:

if $T_1(\omega) = \infty$: $x_t(\omega) = x_t^{i_0}(\omega_{i_0})$, $t \geq 0$
if $T_1(\omega) < \infty$: $x_t(\omega) = x_t^{i_0}(\omega_{i_0})$, $0 \leq t < T_1(\omega)$
$x_{T_1}(\omega)$ is a r.v. according to $\varepsilon_{\omega_{i_0}} \Psi$.

The process restarts from $x_{T_1}(\omega) = x_1^{i_1}$ according to the same recipe, using now the process $(x_t^{i_1})$. Thus if $T_1(\omega) < \infty$ we define the next jump time

$$T_2(\omega) = T_2(\omega)(\omega_{i_0}, \omega_{i_1}) = T_1(\omega_{i_0}) + s_{i_2}(\omega_{i_2}).$$

The sample path $x_t(\omega)$ between the two jump times is now defined as follows:

if $T_2(\omega) = \infty$: $x_t(\omega) = x_{t-T_1}^{i_1}(\omega_{i_1})$, $t \geq T_1(\omega)$
if $T_2(\omega) < \infty$: $x_t(\omega) = x_t^{i_1}(\omega_{i_1})$, $0 \leq T_1(\omega) \leq t < T_2(\omega)$
$x_{T_2}(\omega)$ is a r.v. according to $\varepsilon_{\omega_{i_1}} \Psi$.

Generally, if $T_k(\omega) <$ then the next jump time is

$$T_{k+1}(\omega) = T_{k+1}(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) = T_k(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}) + S^{i_k}(\omega_{i_k}) \tag{2.6}$$

and the sample path $x_t(\omega)$ between the two jump times T_k and T_{k+1} is defined as follows:

$$\begin{array}{lcl} \text{if } T_{k+1}(\omega) & = & \infty: \ x_t(\omega) = x^{i_k}_{t-T_k}(\omega_{i_k}), t \geq T_{k+1}(\omega) \\ \text{if} T_{k+1}(\omega) & < & \infty: \begin{array}{l} x_t(\omega) = x^{i_k}_{t-T_k}(\omega_{i_k}), 0 \leq T_k(\omega) \leq t < T_{k+1}(\omega) \\ x_{T_{k+1}}(\omega) \text{ is a r.v. according to } \varepsilon_{\omega_{i_k}}\Psi. \end{array} \end{array} \tag{2.7}$$

Thus, we have constructed a sequence of times $0 < T_1 < T_2 < ... < T_n < ...$ and we have defined a map from Ω into the set of sequences $(S^{i_1}(\omega), Z_1(\omega), ..., S^{i_{k_0}}(\omega))$, where $k_0 = \min\{k : S^{i_k}(\omega) = \infty\}$ and $Z_k(\omega) = x^{i_k}_{T_k}(\omega)$. Let $T_\infty = \lim_{n\to\infty} T_n$. Then $x_t(\omega) = \Delta$ if $t \geq T_\infty$.

Therefore a sample path until T_{k_0} of the process (x_t), starting from a fixed initial point $x_0 = (i_0, x^{i_0}_0)$, is obtained by the following concatenation:

$$\omega = \omega_{i_0} * \omega_{i_1} * ... * \omega_{i_{k_0-1}}.$$

We denote

$$N_t(\omega) = \sum I_{(t \geq T_k)}$$

Assumption 2 *For every starting point $x \in X$, $EN_t < \infty$, for all $t \in \mathbb{R}_+$.*

2.2.4 Wiener Probabilities

Under the assumption 2, the underlying probability space Ω can be identified with $D_{[0,\infty)}(X)$. Denote by $\widetilde{x}_t$ the coordinate function $\widetilde{x}_t(z) = z(t)$ for $z \in D_{[0,\infty)}(X)$. With assumption 2 in force, the above construction defines for each starting point $x \in X$ a measurable mapping $\phi_x : \Omega \to D_{[0,\infty)}(X)$ such that $\widetilde{x}_t(\phi_x(\omega)) = x_t(\omega)$. Let P_x denote the image measure

$$P_x = P\phi_x^{-1}. \tag{2.8}$$

Thus a Markov string can be thought of as a process defined on Ω or as a Markov family defined on $D_{[0,\infty)}(X)$.

Remark 2 *For each $i \in Q$, there exists a relation, which is similar to (2.8), between P^i and $P^i_{x^i}$, i.e.*

$$P^i_{x^i} = P^i(\phi^i_{x^i})^{-1} \tag{2.9}$$

where $\phi^i_{x^i} : \Omega_i \to D_{[0,\infty)}(X^i)$ is a measurable function such that $\widetilde{x}^i_t(\phi^i_{x^i}(\omega_i)) = x^i_t(\omega_i)$.

Remark 3 *The above construction of a Markov string allows us to write for each starting point $x_0 = x^{i_0}_0$*

$$\phi_{x_0}(\omega) = (\phi^{i_0}_{x^{i_0}_{T_0}}(\omega_{i_0}) * ... * \phi^{i_k}_{x^{i_k}_{T_k}}(\omega_{i_k}) * ...). \tag{2.10}$$

and $t > 0$ is such that $T_k(\omega) \leq t < T_{k+1}(\omega)$ then

$$\widetilde{x}_t(\phi_{x_0}(\omega)) = x^{i_k}_{t-T_k}(\omega_{i_k}) = \widetilde{x}^{i_k}_{t-T_k}(\phi^{i_k}_{x^{i_k}_{T_k}}(\omega_{i_k})).$$

Remark 4 *For each starting point $x \in X$, if A is a measurable set of trajectories in $D_{[0,\infty)}(X)$ then $\phi_x^{-1}(A) = \coprod_{i\in Q} \mathbf{A}^i$ where $\mathbf{A}^i \in \mathcal{F}^i$ and we have*

$$P_x(A) = P\phi_x^{-1}(A) = P(\coprod_{i\in Q} \mathbf{A}^i) = \coprod_{i\in Q} P^i(\mathbf{A}^i)$$

In this case, because the relation (2.10) depends on the post-jump locations, an explicit relation between the probabilities on the trajectories of the whole process and the same probabilities of the component processes can not be written.

Remark 5 *Since any $\omega \in \Omega$ is a right continuous function with left limits defined on X, it can be obtained as the concatenation of such kind of functions defined on the components X^i, i.e. it is possible, for example, to emphases the i-th component of ω*

$$\omega = \omega_i * \omega_i^-.$$

One might define the expectation $E^x f$, $x \in X$, where f is a $\mathcal{F}$-measurable function on Ω, which depends only on a finite number of variables, by recursion on the number of variables.

1) If f depends only on ω_i, i.e. $f(\omega) = f_1(\omega_i)$ with f_1 a $\mathcal{F}^i$-measurable function on Ω^i, then

- if $x = x^i \in X^i$
$$E_x f = E^i_{x^i} f$$
where $E^i_{x^i}$ is the expectation corresponding to the probability $P^i_{x^i}$;
- if $x = x^j \in X^j$, $j \neq i$ then
$$E_x f = 0.$$

2) If f depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_n}$, i.e. $f(\omega) = f_n(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_n})$ with f_n a $\prod_{k=1}^{n} \mathcal{F}^{i_k}$-measurable function on $\prod_{k=1}^{n} \Omega^{i_k}$ then

$$\begin{aligned} f_{n-1}(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{n-1}}) &= \int_{\Omega^{i_n}} f_n(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{n-1}}, \omega_{i_n}) dP^{i_n}_{\Psi(\omega_{i_{n-1}}, \cdot)}(\omega_{i_n}); \\ g(\omega) &= f_{n-1}(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{n-1}}); \\ E_x f &= E_x g. \end{aligned} \tag{2.11}$$

2.2.5 Translation Operators

Let us define now the translation operator (θ_t) associated with (x_t). If $t \geq T_\infty(\omega)$, then we take $\theta_t(\omega) = [\Delta] = ([\Delta]^i)_{i \in Q}$. Otherwise, there exists k such that $T_k(\omega) \leq t < T_{k+1}(\omega)$. In this case we take

$$\theta_t(\omega) = (\theta^{i_k}_{t-T_k(\omega)}(\omega_{i_k}), \omega_{i_{k+1}}, ...). \tag{2.12}$$

Lemma 1 *(θ_t) is the translation operator associated with (x_t), i.e.*

$$\theta_s \circ \theta_t = \theta_{s+t}; \;\; x_s \circ \theta_t = x_{s+t}.$$

Proof. If $t \geq T_\infty(\omega)$, then $\theta_t(\omega) = [\Delta]$ and $x_{s+t}(\omega) = \Delta = x_s(\theta_t(\omega))$.
Suppose that there exist $k, i \geq 0$ such that $T_k(\omega) \leq t < T_{k+1}(\omega)$ and $T_i(\theta_t\omega) \leq s < T_{i+1}(\theta_t\omega)$. Then

$$\begin{aligned} x_t(\omega) &= x^{i_k}_{t-T_k}(\omega_{i_k}); \\ (x_s \circ \theta_t)(\omega) &= x^{i_l}_{s-T_l}(\theta^{i_l}_{s-T_l}\omega_{i_l}). \end{aligned}$$

Since $\theta_t(\omega)$ is given by (2.12) and T_{k+1} is given by (2.6) we obtain

$$\begin{aligned} T_{k+1}(\theta_t\omega) &= S^{i_k}(\theta^{i_k}_{t-T_k(\omega)}(\omega_{i_k})) = \\ &= S^{i_k}(\omega_{i_k}) - (t - T_k(\omega)) = \\ &= T_{k+1}(\omega) - t. \end{aligned}$$

Then
$$T_{i+1}(\theta_t\omega) = T_{k+i+1}(\omega) - t$$
Therefore
$$T_i(\theta_t\omega) \leq s < T_{i+1}(\theta_t\omega) \Leftrightarrow T_{k+i}(\omega) \leq s + t < T_{k+i+1}(\omega).$$

2.2.6 Markov Chain

Let (p_n) be a discrete time Markov chain associated to (x_t) with the state space $(\bigcup_{i \in Q} \Omega_i, \widehat{\mathcal{F}})$ and the underlying probability space $(\Omega, \mathcal{F})$. The chain (p_n) is essentially 'the $n-th$' step of the process (x_t). If its starting point is ω_{i_0} (a trajectory in Ω^{i_0} starting in $x^{i_0}_0$) then $p_n(\omega) = \omega_{i_n}$.

The transition kernel associated with (p_n) can be defined using the kernel Ψ as follows:

$$H(\widehat{\omega}, A) = P_{\varepsilon_{\widehat{\omega}}\Psi}(A), \;\; A \in \widehat{\mathcal{F}}.$$

The construction of P^x from subsection 2.2.4 is such that

- H is the transition function of (p_n);

- P^x is the initial probability law of (p_n); i.e. if $\widehat{\omega} \in \bigcup_{i \in Q} \Omega_i$ which starts in $x \in X$

$$P^{\widehat{\omega}}(p_0 \in A) = P^x(A),\ A \in \mathcal{F}.$$

Let η_k the projection $(p_0, p_1, ..., p_k)$, i.e.

$$\eta_k(\omega) = (\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}).$$

2.2.7 Natural Filtrations

Let $(\mathcal{F}_t)$ be the natural filtration with respect to (x_t). The natural filtration $(\mathcal{F}_t)$ on Ω is built such that we have the following definition of $\mathcal{F}_t$-measurability:

Definition 2 *A $\mathcal{F}$-measurable function f on Ω is $\mathcal{F}_t$-measurable if the following property holds:*

For each k, the function $f \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}$ is equal to $h \circ \eta_k$, where the function $h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k})$ is such that for each fixed $(\widehat{\omega}_{i_0}, \widehat{\omega}_{i_1}, ..., \widehat{\omega}_{i_{k-1}})$ with $T_k(\widehat{\omega}_{i_0}, \widehat{\omega}_{i_1}, ..., \widehat{\omega}_{i_{k-1}}) \leq t$, $\omega_{i_k} \mapsto h(\widehat{\omega}_{i_0}, \widehat{\omega}_{i_1}, ..., \widehat{\omega}_{i_{k-1}}, \omega_{i_k})$ is measurable with respect to $\mathcal{F}^{i_k}_{t-T_k}$.

Because the families of filtrations $(\mathcal{F}^i_t)$ are nondecreasing and right continuous one can verify that the family $(\mathcal{F}_t)$ has the same properties, as follows.

Proposition 3 *(i) The family $(\mathcal{F}_t)$ is nondecreasing and right continuous.*

(ii) The random variables T_k are stopping times with respect to $(\mathcal{F}_t)$.

(iii) Let T a stopping time with respect to $(\mathcal{F}_t)$. For each $k \in \mathbb{N}$, $T \wedge T_k$ is a function on Ω which depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}$. On the other hand, if $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}$ are fixed, the function $(T \wedge T_{k+1} - T_k)^+$ with ω_{i_k} as argument is a stopping time with respect $(\mathcal{F}^{i_k}_t)$.

Proof. (i) We have to prove that $t \leq u$ implies $\mathcal{F}_t \subset \mathcal{F}_u$. Let f be a $\mathcal{F}_t$-measurable function. Then

$$f = \sum f \cdot I_{\{T_k \leq t < T_{k+1}\}} + f \cdot I_{\{T_\infty \leq t\}}. \tag{2.13}$$

Since each function which appears in right-hand side of (2.13) is $\mathcal{F}_t$-measurable, it remains to show that each one is also $\mathcal{F}_u$-measurable. This is obvious for $f \cdot I_{\{T_\infty \leq t\}}$ since $f \cdot I_{\{T_\infty \leq t\}} \cdot I_{\{T_n \leq u < T_{n+1}\}} = 0, \forall n$. Similarly, $f \cdot I_{\{T_k \leq t < T_{k+1}\}} \cdot I_{\{T_n \leq u < T_{n+1}\}} = 0, \forall n < k$. If $n = k$ then $f \cdot I_{\{T_n \leq u < T_{n+1}\}}$ depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_n}$. If $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{n-1}}$ are fixed then the partial function in ω_{i_n} is $\mathcal{F}^{i_n}_{t-T_k}$-measurable, therefore it is $\mathcal{F}^{i_n}_{u-T_k}$-measurable.
Finally, if $n > k$ then $f \cdot I_{\{T_k \leq t < T_{k+1}\}}$ depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}$ (thus on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_n}$) and for fixed $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{n-1}}$ the corresponding application is constant.
Let us prove now that $(\mathcal{F}_t)$ is right continuous. Suppose that f is $\mathcal{F}_{t+\varepsilon}$-measurable, for each $\varepsilon > 0$. We have

$$f \cdot I_{\{T_k \leq t < T_{k+1}\}} = \lim_{\varepsilon \to 0+} f \cdot I_{\{T_k \leq t+\varepsilon < T_{k+1}\}}.$$

The functions in the right-hand side, for each $\varepsilon > 0$, depend only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}$. For fixed $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}$ the partial application of the left-hand side in ω_{i_k} is measurable with respect to $\bigcap_{\varepsilon>0} \mathcal{F}^{i_k}_{t+\varepsilon-T_k}$ which is equal to $\mathcal{F}^{i_k}_{t-T_k}$ since the family $(\mathcal{F}^{i_k}_t)$ is right continuous. Therefore, f is $\mathcal{F}_t$-measurable.
(ii) We have

$$\{T_k \leq t\} \cap \{T_n \leq t < T_{n+1}\} = \begin{cases} \emptyset & \text{if } n \leq k; \\ \{T_n \leq t < T_{n+1}\} & \text{if } n > k. \end{cases}$$

Since the indicator function of $\{T_k \leq t\}$ satisfies the definition 2, T_k is indeed a stopping time with respect to $(\mathcal{F}_t)$.
Because each stopping time is the inferior envelope of a step stopping time sequence, we can suppose without loss of generality that

$$T(\omega) = \begin{cases} t & \text{if } \omega \in A \\ +\infty & \text{if } \omega \notin A \end{cases}$$

where $A \in \mathcal{F}_t$. Let $B = A \cap \{t < T_k\}$. We get

$$T \wedge T_k(\omega) = \begin{cases} t & \text{if } \omega \in B \\ T_k(\omega) & \text{if } \omega \notin B \end{cases}$$

We have

$$B = \bigcup_{n \leq k} A \cap \{t < T_k\} \cap \{T_{n-1} \leq t < T_n\},$$

Since the indicator function $I_{\{t<T_k\}}$ depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}$ and $I_{A\cap\{T_{n-1}\leq t<T_n\}}$ depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_n}$ we get that I_B, so $T \wedge T_k$, depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}$.
Let $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}$ be fixed and let us denote by u the partial application $(T \wedge T_{k+1} - T_k)^+$ of ω_{i_k}. If $T_k(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}) > t$ then $u = 0$. Otherwise, let h be the indicator function of $A \cap \{T_k \leq t < T_{k+1}\}$ which depends only on $\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}$. Let

$$D = \{\omega_{i_k} \in \Omega^{i_k} | h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) = 1\}$$

which belongs to $\mathcal{F}^{i_k}_{t-T_k}$ (because $A \in \mathcal{F}_t$). On the other hand we have $u(\omega_{i_k}) = S^{i_k}(\omega_{i_k}) \wedge v(\omega_{i_k})$ where the function v given by

$$v(\omega_{i_k}) = \begin{cases} t - T_k & \text{if } \omega_{i_k} \in D \\ +\infty & \text{if } \omega_{i_k} \notin D \end{cases}$$

is a stopping time. It results that u is also a stopping time with respect to $\mathcal{F}^{i_k}_t$.

2.2.8 Jump Process

Fix $x \in X$ and consider a Markov string (x_t) starting at x as constructed above. The associated *jump process* (η_t) takes values in $X \times \mathbb{Z}_+$ and is defined as

$$\eta_t = \begin{bmatrix} x \\ 0 \end{bmatrix}, t < T_1, ..., \eta_t = \begin{bmatrix} \eta^1_t \\ \eta^2_t \end{bmatrix} = \begin{bmatrix} x_{T_k} \\ k \end{bmatrix}, T_k \leq t < T_{k+1}.$$

We do not have a one-to-one correspondence between the sample paths of (x_t) and (η_t) as in the case of piecewise deterministic Markov processes [12]. Given the sample path $\{x_s,\ s \leq t\}$, the jump times the finite set $\{T_j,\ j = 1, ..., k\} = \{s \in (0, t] : x_s \neq x_{s-}\}$, and the sample path $\{\eta_s,\ s \leq t\}$ is defined using the above recipe. But conversely, given $\{\eta_s,\ s \leq t\}$ we know that $x_0 = \eta^1_0$ and it is possible to find more then one trajectories which start from x_0 and so on. In fact, to a sample trajectory of (η_t) we might associate with it a set of sample paths of (x_t).
Therefore, the above jump process will not serve us in the studying of Markov string as in [12]. Its role will be taken by the Markov chain constructed in section 2.2.6.

2.3 Basic Properties

2.3.1 Simple Markov Property

Mainly, in this section we prove that the Markov string (x_t) constructed in section 2.2.1 is a right Markov process. The proof engine is based on the Markov property of the discrete time Markov chain (p_n).

Remark 6 *For each k on the set $\{T_k(\omega) \leq t < T_{k+1}(\omega)\}$ we have*

$$x_t = x^{i_k}_{t-T_k} \circ p_k$$

Proposition 4 *Any Markov string $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$, obtained using the procedure presented in section 2.2.1, is a Markov process.*

Proof. The simple Markov property of (x_t) is equivalent to the following implication [21]:
If f is a positive $\mathcal{F}_t$-measurable function and g is a $\mathcal{F}$-measurable function then

$$E^x[f \cdot g \circ \theta_t] = E^x[f \cdot E^{x_t}[g]]. \tag{2.14}$$

The identity (2.14) can be unfolded into two separated equalities

$$E^x[f \cdot g \circ \theta_t \cdot I_{\{t \geq T_\infty\}}] = E^x[f \cdot E^{x_t}[g] \cdot I_{\{t \geq T_\infty\}}] \tag{2.15}$$

$$E^x[f \cdot g \circ \theta_t \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}] = E^x[f \cdot E^{x_t}[g] \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}] \tag{2.16}$$

The identity (2.15) is clear because on $\{t \geq T_\infty\}$

$$\begin{aligned} E^{x_t}[g] &= g([\Delta]); \\ \theta_t(\omega) &= [\Delta]; \\ x_t(\omega) &= \Delta. \end{aligned}$$

Let us prove now the identity (2.16). Let $\omega \in \Omega$. By the definition of $\mathcal{F}_t$ we have

$$f(\omega) \cdot I_{\{T_k(\omega) \le t < T_{k+1}(\omega)\}}(\omega) = h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) \tag{2.17}$$

where h is a measurable function as in the definition 2 and is equal to zero outside of the set $\{T_k(\omega) \le t < T_{k+1}(\omega)\}$.

In order to prove (2.16) it is enough to treat the case when the function g depends only on a finite number of variables (because the expectation E^x is defined by the recursion (2.11)).

We start with the case when the function g depends only on a single variable, ω_{i_0}, i.e.

$$g(\omega) = a(\omega_{i_0})$$

where a is $\mathcal{F}^{i_0}$-measurable on Ω^{i_0}. In this case, the left-hand side of (2.16) is equal to

$$E^x[f \cdot I_{\{T_k(\omega) \le t < T_{k+1}(\omega)\}} \cdot a(\theta^{i_k}_{t-T_k(\omega)}(\omega_{i_k}))]. \tag{2.18}$$

Because the term between [...] depends only on $(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k})$ we obtain that the (2.18) becomes

$$E^x\{\int_{\Omega^{i_k}} h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) \cdot a(\theta^{i_k}_{t-T_k(\omega)}(\omega_{i_k}))dP^{i_k}_{\Psi(\omega_{i_{k-1}},\cdot)}(\omega_{i_k})\}. \tag{2.19}$$

Again, the integrand between $\{...\}$ depends only on $(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}})$. Since the function $\omega_{i_k} \to h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k})$ is $\mathcal{F}^{i_k}_{t-T_k}$-measurable, we can use the Markov property of the process $\mathbb{M}^{i_k}$ and (2.19) becomes

$$\int_{\Omega^{i_k}} h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) E^{i_k}_{x^{i_k}_{t-T_k}(\omega_{i_k})}[a]dP^{i_k}_{\Psi(\omega_{i_{k-1}},\cdot)}(\omega_{i_k})\}. \tag{2.20}$$

Since $x_t(\omega) = x^{i_k}_{t-T_k}(\omega_{i_k})$ on $\{T_k(\omega) \le t < T_{k+1}(\omega)\}$ the computation of the right-hand side of (2.16) gives

$$E^x\{h(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k}) \cdot E^{i_k}_{x^{i_k}_{t-T_k}(\omega_{i_k})}[a]\} \tag{2.21}$$

Using the recursive procedure, as before, (2.21) gives (2.20).

Suppose now that (2.16) is established for all functions g which depend only on $(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}})$. We have to prove that (2.16) is true for

$$g(\omega) = g(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k});\ k > 0.$$

Let

$$c(\omega) = c(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_{k-1}}) = \int_{\Omega^{i_k}} b(\omega_{i_0}, \omega_{i_1}, ..., \omega_{i_k})dP^{i_k}_{\Psi(\omega_{i_{k-1}},\cdot)}(\omega_{i_k}).$$

Using the recursive procedure, one can check that the functions

$$h(...)g \circ \theta_t \quad \text{and} \quad h(...)c \circ \theta_t$$

have the same expectations.
On the other hand, the functions

$$h(...)E_{x_t}[g] \quad \text{and} \quad h(...)E_{x_t}c$$

have the same expectations. Since c depends only on $k-1$ variables, this implies (2.16) for the general case.

2.3.2 Cadlag Property

Proposition 5 *If $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$ is a Markov string as in section 2.2.1, then for all $\omega \in \Omega$ the trajectories $t \mapsto x_t(\omega)$ are right continuous on $[0, \infty)$ with left limits on $(0, \infty)$.*

Proof. The result is a direct consequence of two facts:

1. the sample paths of (x_t) are obtained by concatenation of sample paths of component process;
2. the component processes enjoy the càdlàg property.

Then the Markov string inherits the càdlàg property.

2.3.3 Strong Markov Property

Theorem 6 *Any Markov string $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$, obtained using the procedure presented in section 2.2.1, is a strong Markov process.*

Each T_k is a stopping time for (x_t) (see proposition 3 (ii)). For each $k \geq 1$, T_k can be obtained by the following recursion

$$T_{k+1} = T_k + S^{i_k} \circ \theta_{T_k}$$

Let us prove now that the process (x_t) is a strong Markov process. The filtration $(\mathcal{F}_t)$ is nondecreasing and right continuous (see proposition 3 (i)). Then the process (x_t) satisfies the right hypothesis.

Let (P_t) be the semigroup of the whole Markov process (x_t), $P_t g(x) = E_x g(x_t)$, where g is bounded $\mathcal{B}$-measurable function. Let $(U_p)_{p>0}$ the resolvent associated to the semigroup, i.e.

$$U_p g = \int_0^\infty e^{-pt} P_t g dt.$$

It is known that the strong Markov property is equivalent with each from the following assertions [20]:

1. If g is a positive bounded continuous function on X_Δ then $f = U_p g$ $(p > 0)$ is nearly Borel and right continuous on the process trajectories.
2. Each p-excessive function $(p > 0)$ is nearly Borel and right continuous on the process trajectories.

Recall that a real function defined on the state space X_Δ is nearly Borel for the process (x_t) if there exist two Borel function h and h' on X_Δ such that $h' \leq f \leq h$ and

$$P\{\omega | \exists t, h' \circ x_t(\omega) < h \circ x_t(\omega)\} = 0. \tag{2.22}$$

Let g be a positive bounded continuous function on X. We have $g = \sum_{i \in Q} g^i$, where $g^i = g|_{X^i}$ are bounded continuous functions on X^i. Then $P_t g = \sum_{i \in Q} P_t^i g^i$ and

$$\begin{aligned} U_p g &= \int_0^\infty e^{-pt} P_t g dt = \\ &= \sum_{i \in Q} \int_0^\infty e^{-pt} P_t^i g^i dt = \\ &= \sum_{i \in Q} U_p^i g^i. \end{aligned}$$

It is known that $f = U_p g$ $(p > 0)$ (the restriction to X) is p-excessive function with respect to (P_t) and for each $i \in Q$ and the function $f^i = U_p^i g^i$ is p-excessive function with respect to (P_t^i). Therefore, f^i is nearly Borel and right continuous on the trajectories of the process (x_t^i). It is clear from the construction that the function f is right continuous on the trajectories of the process (x_t).

Let $h^i, h^{i\prime}$ two Borel functions on X_Δ^i such that $h' \leq f^i \leq h^i$ and

$$h^{i\prime} \circ x_t^i(\omega_i) = h^i \circ x_t^i(\omega_i) \ P^i - a.s., \forall t \geq 0. \tag{2.23}$$

Let us consider the function h, h' defined as below:

$$h = \sum_{i \in Q} h^i, \ h' = \sum_{i \in Q} h^{i\prime}. \tag{2.24}$$

It is clear that

$$P\{\omega | \exists t \geq T_\infty, h' \circ x_t(\omega) < h \circ x_t(\omega)\} = 0.$$

Let us compute the probability of the following event:

$$A_k = \{\exists t | T_k \leq t < T_{k+1}, h' \circ x_t(\omega) < h \circ x_t(\omega)\}.$$

We have $A_k \in \mathcal{F}$. Let $a_k = I_{A_k}$ which depends only on $\omega_{i_0}, \omega_{i_2}, ..., \omega_{i_k}$. Using the recursive method to compute the probability of A_k on $\{T_k \leq t < T_{k+1}\}$, we obtain

$$\int_{\Omega^{i_k}} a_k(\omega_{i_0}, \omega_{i_2}, ..., \omega_{i_k}) dP^{i_k}_{\Psi(\omega_{i_{k-1}}, \cdot)}(\omega_{i_k}). \tag{2.25}$$

Since the function $a_k(\omega_{i_0}, \omega_{i_2}, ..., \omega_{i_k})$ on Ω^{i_k} is exactly the indicator function of

$$B = \{\omega_{i_k} | \exists u < S^{i_k}(\omega_{i_k}), h^{i_k\prime} \circ x_u^{i_k}(\omega) < h^{i_k} \circ x_u^{i_k}(\omega)\}$$

using (2.23) we obtain that the integral (2.25) is zero. Therefore the functions h, h' defined by (2.24) verify the condition (2.22). Then f will be a nearly Borel function relative to the process (x_t).

Chapter 3 General Stochastic Hybrid Processes

3.1 Description

General Stochastic Hybrid Models (GSHM) are a class of non-linear stochastic continuous-time hybrid dynamical systems. GSHM are characterized by an hybrid state defined by two components: the continuous state and the discrete state. The continuous state evolves in according to a SDE whose vector field and drift factor depend on the hybrid state, both continuous and discrete. Switching between two discrete states is governed by a probability law or occurs when the continuous state hits the boundary of its state space. Whenever a switching occurs, the hybrid state is reset instantly to a new state in according to a probability law which depends itself on the past hybrid state.

GSHM involve a hybrid state space, with both continuous and discrete states. The continuous and the discrete parts of the state variable have their own natural dynamics, but the main point is to capture the interaction between them.

The time t is measured continuously. The state of the system is represented by a continuous variable x and a discrete variable i. The continuous variable evolves in some "cells" X^i (open sets in the Euclidean space) and the discrete variable belongs to a countable set Q. The intrinsic difference between the discrete and continuous variables, consists of the way that they evolve through time. The continuous state is governed by an SDE that depends on the hybrid state. The discrete dynamics produces transitions in both (continuous and discrete) state variables x, i. Transitions occur when the continuous state hits the boundary of the state space (forced transitions) or according with a probability law. Whenever a transition occurs the hybrid state is reset instantly to a new value. The value of the discrete state after the transition is determined by the hybrid state before the transition. On the other hand, the new value of the continuous state obeys a probability law which depends on the last hybrid state. Thus, a sample trajectory has the form $(q_t, x_t, t \geq 0)$, where $(x_t, t \geq 0)$ is piecewise continuous and $q_t \in Q$ is piecewise constant. Let $(0 \leq T_1 \leq T_2 \leq ... \leq T_i \leq T_{i+1} \leq ...)$ be the sequence of jump times at which the continuous and the discrete part of the system interact. This time sequence is generated when the state of the system hits the boundary or according with a transition rate.

In fact, GSHM are a class of stochastic processes which generalizes the models introduced by Davis in [12]. The difference between GSHM and PDMP is that for GSHM between two consecutive jumps the process is a diffusion whilst for PDMP the inter-jumps motion is deterministic, according to a vector field.

3.2 The Abstract Model

3.2.1 State space

Let Q be a countable set of discrete states, and let $d : Q \to \mathbb{N}$ and $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ be two maps assigning to each discrete state $i \in Q$ an open subset X^i of $\mathbb{R}^{d(i)}$. We call the set

$$X(Q, d, \mathcal{X}) = \bigcup_{i \in Q} \{i\} \times X^i$$

the hybrid state space of the GSHM and $x = (i, x^i) \in X(Q, d, \mathcal{X})$ the hybrid state. The completion of the hybrid state space will be

$$\overline{X} = X \cup \partial X$$

where

$$\partial X = \bigcup_{i \in Q} \{i\} \times \partial X^i,$$

It is clear that, for each $i \in Q$, the state space X^i is a Borel space. It is possible to define a metric ρ on X in such a way the restriction of ρ to any component X^i is equivalent to the usual Euclidean metric (see chapter 2). Then $(X, \mathcal{B}(X))$ is a Borel space (see chapter 2 for the construction of $\mathcal{B}(X)$). Moreover, X is a homeomorphic with a Borel subset of a compact metric space (Lusin space) because it is a locally compact Hausdorff space with countable base (see [12] and the references therein).

3.2.2 Construction

Assumption 3 *Suppose that $b : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)}$, $\sigma : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot) \times m}$, $m \in \mathbb{N}$, are bounded and Lipschitz continuous in x.*

This assumption ensures, for any $i \in Q$, the existence and uniqueness (Theorem 6.2.2. in [2]) of the solution for the following stochastic differential equation (SDE)

$$dx(t) = b(i, x(t))dt + \sigma(i, x(t))dW_t, \tag{3.1}$$

where $(W_t, t \geq 0)$ is the m-dimensional standard Wiener process in a complete probability space.

In this way, when i runs in Q, the equation (3.1) defines a family of diffusion processes $\mathbb{M}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, x^i_t, \theta^i_t, P^i)$, $i \in Q$ with the state spaces $\mathbb{R}^{d(i)}$, $i \in Q$.

The jump (switching) mechanism between the diffusions is governed by two functions: the jump rate λ and the transition measure R. The jump rate $\lambda : X \to \mathbb{R}_+$ is a measurable function and the transition measure R maps X into the set $\mathcal{P}(X)$ of probability measure on $(X, \mathcal{B}(X))$.

One can consider the transition measure $R : \overline{X} \times \mathcal{B}(X) \to [0,1]$ as a reset probability kernel such that: (i) for all $A \in \mathcal{B}(X)$, $R(\cdot, A)$ is measurable; (ii) for all $x \in \overline{X}$ the function $R(x, \cdot)$ is a probability measure.

Assumption 4 *(i) $\lambda : X \to \mathbb{R}_+$ is a measurable function such that $t \to \lambda(x^i_t(\omega_i))$ is integrable on $[0, \varepsilon(x^i))$, for some $\varepsilon(x^i) > 0$, for each $x^i \in X^i$ and each ω_i starting at x^i.*

(ii) $R(x, \{x\}) = 0$ for $x \in X$.

Since $\overline{X}$ is a Borel space, then $\overline{X}$ is homeomorphic to a subset of the Hilbert cube, $\mathcal{H}$[1] (Urysohn's theorem, Prop. 7.2 [5]). Therefore, its space of probabilities is homeomorphic to the space of probabilities of the corresponding subset of $\mathcal{H}$ (Lemma 7.10 [5]). There exists a measurable function $F : \mathcal{H} \times \overline{X} \to X$ such that

$$R(x, A) = \mathfrak{p}F^{-1}(A),\ A \in \mathcal{B}(X) \tag{3.2}$$

where $\mathfrak{p}$ is the probability measure on $\mathcal{H}$ associated to $R(x, \cdot)$ and

$$F^{-1}(A) = \{\boldsymbol{\omega} \in \mathcal{H} | F(\boldsymbol{\omega}, x) \in A\}.$$

The measurability of such a function is guaranteed by the measurability properties of the transition measure R.

We construct an GSHM as a *Markov string* H which admits $(\mathbb{M}^i)$ as subprocesses. The sample path of the stochastic process $(x_t)_{t>0}$ with values in X, starting from a fixed initial point $x_0 = (i_0, x^{i_0}_0) \in X$ is defined as in chapter 2, section 2.2.1 using a particular sequence of stopping times and a particular renewal kernel. We have to precise, from the beginning, that the above recipe gives a sample path of GSHM starting with a initial diffusion path whose starting point is x_0. An arbitrary point x_0 does not define in a unique way a diffusion path!

Let ω_i a trajectory which starts in (i, x^i). Let $t_*(\omega_i)$ be the first hitting time of ∂X^i of the process (x^i_t). Let us define the function

$$F(t, \omega_i) = I_{(t < t_*(\omega_i))} \exp(-\int_0^t \lambda(i, x^i_s(\omega_i)))ds. \tag{3.3}$$

This function will be the survivor function for the stopping time S^i associated to the diffusions (x^i_t), which will be employed in the construction of our string. This means that the stopping time S^i satisfies the condition

$$P^i(S^i > t) = F(t, \omega_{i_0}).$$

Obviously, the stopping time S^i is the minimum of two other stopping times:

1. first hitting time of boundary, i.e. $t_*|_{\Omega^i}$;
2. the stopping time $S^{i\prime}$ which satisfies $P^i(S^{i\prime} > t) = \exp(-\int_0^t \lambda(i, x^i_s(\omega_i)))ds$.

The first jump time of the process $T_1(\omega) = T_1(\omega_{i_0}) = S^{i_0}(\omega_{i_0})$. The sample path $x_t(\omega)$ up to the first jump time is now defined as follows:

if $T_1(\omega) = \infty :\quad x_t(\omega) = (i_0, x^{i_0}_t(\omega_{i_0}))$, $t \geq 0$

if $T_1(\omega) < \infty :\quad x_t(\omega) = (i_0, x^{i_0}_t(\omega_{i_0}))$, $0 \leq t < T_1(\omega)$

$\qquad x_{T_1}(\omega) = F(\boldsymbol{\omega}, (i_0, x^{i_0}_{T_1}(\omega_{i_0})))$.

[1] $\mathcal{H}$ is the product of countable many copies of $[0, 1]$.

The process restarts from $x_{T_1}(\omega) = (i_1, x_1^{i_1})$ according to the same recipe, using now the process $x_t^{i_1}$. Thus if $T_1(\omega) < \infty$ we define the next jump time

$$T_2(\omega) = T_2(\omega_{i_0}, \omega_{i_1}) = T_1(\omega_{i_0}) + S^{i_1}(\omega_{i_1})$$

The sample path $x_t(\omega)$ between the two jump times is now defined as follows:

if $T_2(\omega) = \infty$: $\quad x_t(\omega) = (i_1, x_{t-T_1}^{i_1}(\omega))$, $t \geq T_1(\omega)$

if $T_2(\omega) < \infty$: $\quad x_t(\omega) = (i_1, x_t^{i_1}(\omega))$, $0 \leq T_1(\omega) \leq t < T_2(\omega)$

$\quad x_{T_2}(\omega) = F(\boldsymbol{\omega}, (i_1, x_{T_2}^{i_1}(\omega)))$.

and so on.

Let $T_1 < T_2 < ... < T_n < ...$ be the sequence of stopping times obtained by the above method. Let $T_\infty = \lim_{n\to\infty} T_n$.

All standard probabilistic elements associated to (x_t) are constructed as in the section 2.2.1. As well, we suppose that the assumption 2 is in force.

3.2.3 Formal Definitions

We can introduce the following definition.

Definition 7 *A General Stochastic Hybrid Model (GSHM) is a collection* $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$ *where*

- Q *is a countable set of discrete variables;*
- $d : Q \to \mathbb{N}$ *is a map giving the dimensions of the continuous state spaces;*
- $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ *maps each* $q \in Q$ *into an open subset* X^q *of* $\mathbb{R}^{d(q)}$*;*
- $b : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)}$ *is a vector field;*
- $\sigma : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)\times m}$ *is a* $X^{(\cdot)}$*-valued matrix,* $m \in \mathbb{N}$*;*
- $Init : \mathcal{B}(X) \to [0, 1]$ *is an initial probability measure on* $(X, \mathcal{B}(S))$*;*
- $\lambda : \overline{X}(Q, d, \mathcal{X}) \to \mathbb{R}^+$ *is a transition rate function;*
- $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0, 1]$ *is a transition measure.*

We can introduce the GSHM execution.

Definition 8 (GSHM Execution) *A stochastic process* $x_t = (q(t), x(t))$ *is called a GSHM execution if there exists a sequence of stopping times* $T_0 = 0 \leq T_1 \leq T_2 \leq \ldots$ *such that for each* $k \in \mathbb{N}$*,*

- $x_0 = (q_0, x_0^{q_0})$ *is a* $Q \times X$*-valued random variable extracted according to the probability measure* $Init$*;*
- *For* $t \in [T_k, T_{k+1})$*,* $q_t = q_{T_k}$ *is constant and* $x(t)$ *is a (continuous) solution of the SDE:*

$$dx(t) = b(q_{T_k}, x(t))dt + \sigma(q_{T_k}, x(t))dW_t \tag{3.4}$$

  where W_t *is a the* m*-dimensional standard Wiener;*
- $T_{k+1} = T_k + S^{i_k}$ *where* S^{i_k} *is chosen according with the survivor function (3.3).*
- *The probability distribution of* $x(T_{k+1})$ *is governed by the law* $R\left((q_{T_k}, x(T_{k+1}^-)), \cdot\right)$.

3.3 Properties

Notations. Given a function $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and a vector field $b : \mathbb{R}^n \to \mathbb{R}^n$, we use $\mathcal{L}_b f$ to denote the Lie derivative of f along b, i.e. $\mathcal{L}_b f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) f_i(x)$. Given a function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, we use $\mathbb{H}^f$ to denote the Hamiltonian operator applied to f, i.e. $\mathbb{H}^f(x) = (h_{ij}(x))_{i,j=1\ldots n} \in \mathbb{R}^{n\times n}$, where $h_{ij}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$. Given a matrix $A = (a_{ij})_{i,j=1\ldots n} \in \mathbb{R}^{n\times m}$, A^T denotes the transpose matrix of A and $Tr(A)$ denotes its trace, i.e. $Tr(A) = \sum_{i=1}^n a_{ii}$.

3.3.1 Strong Markov Property

Proposition 9 *Any General Stochastic Hybrid Model H, under the standard assumptions of section 3.2, is a Borel right process.*

Proof. In order to apply the theorem 4 to prove that H is a right Markov process, we have to verify the hypothesis of this theorem. We can suppose without loss of generality that $\Omega^i \cap \Omega^j = \emptyset$. Then, the kernel Ψ can be defined as follows

$$\Psi : \{\bigcup_{i \in Q} \Omega^i\} \times \mathcal{B}(X) \to [0,1]$$

such that

$$\Psi(\omega_i, A) = R(x^i_{S^i(\omega_i)}, A)$$

We need to check that: If $0 < t < S^i(\omega_i)$ then $\Psi(\theta^i_t \omega_i, \cdot) = \Psi(\omega_i, \cdot)$, i.e. the 'memoryless' of the stopping times (S^i)

$$R(x^i_{S^i(\theta^i_t \omega_i)}, \cdot) = R(x^i_{S^i(\omega_i)}, \cdot).$$

In fact, we have to prove that, if $0 < t < t+s < S^i(\omega_i)$

$$P^{x^i}(S^i > t+s | S^i > t) = P^{x^i_t}(S^i > s) \tag{3.5}$$

Using the survivor function defined by (3.3), the left hand side of (3.5) becomes

$$\begin{aligned}
P^{x^i}(S^i &> t+s | S^i > t) = \frac{F(t+s, x^i)}{F(t, x^i)} = \\
&= \frac{I_{\{t+s < t_*(\omega_i)\}} \exp(-\int_0^{t+s} \lambda(x^i_\tau(\omega_i)) d\tau)}{I_{\{t < t_*(\omega_i)\}} \exp(-\int_0^{t} \lambda(x^i_\tau(\omega_i)) d\tau)} = \\
&= I_{\{t+s < t_*(\omega_i)\}} \exp(-\int_t^{t+s} \lambda(x^i_\tau(\omega_i)) d\tau) = \\
&= I_{\{t+s < t_*(\omega_i)\}} \exp(-\int_0^{s} \lambda(x^i_{\tau+t}(\omega_i)) d\tau) = \\
&= I_{\{t+s < t_*(\omega_i)\}} \exp(-\int_0^{s} \lambda(x^i_\tau \circ \theta^i_t(\omega_i)) d\tau)
\end{aligned}$$

The right hand side of (3.5) is

$$P^{x^i_t}(S^i > s) = I_{\{s < t_*(\theta^i_t \omega_i)\}} \exp(-\int_0^{s} \lambda(x^i_\tau \circ \theta^i_t(\omega_i)) d\tau)$$

Since $t^*(\theta^i_t \omega_i) = t_*(\omega_i) - t$ we get $t + s < t_*(\omega_i) \Leftrightarrow s < t_*(\theta^i_t \omega_i)$ and (3.5) is proved.

Therefore, H is a Markov string obtained by mixing some diffusion processes. Moreover, since the state space is a Lusin space, H is a Borel right process.

3.3.2 The Process Generator

We denote by $\mathcal{B}_b(X)$ the set of all bounded measurable functions $f : X \to \mathbb{R}$. This is a Banach space under the norm

$$\|f\| = \sup_{x \in X} |f(x)|$$

Let (P_t) be the semigroup of the whole Markov process (x_t),

$$P_t f(x) = E_x f(x_t) = E_x\{f(x_t) | t < \zeta\}$$

where g is bounded $\mathcal{B}$-measurable function and ζ is the lifetime when the process retires to Δ, i.e.

$$\zeta := \inf\{t | x_t = \Delta\}.$$

Associated with the semigroup (P_t) is its *strong generator* which, loosely speaking, is the derivative of P_t at $t = 0$. Let $D(L) \subset \mathcal{B}_b(X)$ be the set of functions f for which the following limit exists

$$\lim_{t \searrow 0} \frac{1}{t}(P_t f - f) \tag{3.6}$$

and denote this limit Lf. The limit refers to convergence in the norm $\|\cdot\|$, i.e. for $f \in D(L)$ we have

$$\lim_{t \searrow 0} ||\frac{1}{t}(P_t f - f) - Lf|| = 0.$$

Specifying the domain $D(L)$ is an essential part of specifying the operator L.

Let $\mathcal{B}_0$ be the subset of $\mathcal{B}_b(X)$ consisting of those functions f for which $\lim_{t \searrow 0} ||P_t f - f|| = 0$. The semigroup is said to be *strongly continuous* on $\mathcal{B}_0$. $\mathcal{B}_0$ is a closed linear subspace of $\mathcal{B}_b(X)$.

Proposition 10 (martingale property) *[12] For $f \in D(L)$ we define the real-valued process $(C_t^f)_{t \geq 0}$ by*

$$C_t^f = f(x_t) - f(x_0) - \int_0^t Lf(x_s)ds. \tag{3.7}$$

Then for any $x \in X$, the process $(C_t^f)_{t \geq 0}$ is a martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$.

There may be other functions f, not in $D(L)$, for which something akin to (3.7) is still true. In this way we get the notion of *extended generator* of the process.

Let $D(\widehat{L})$ denote the set of measurable functions $f : X \to \mathbb{R}$ with the following property: there exists a measurable function $h : X \to \mathbb{R}$ such that the function $t \to h(x_t)$ is integrable $P_x - a.s.$ for each $x \in X$ and the process

$$C_t^f = f(x_t) - f(x_0) - \int_0^t h(x_s)ds$$

is a local martingale. Then we write $h = \widehat{L}f$ and call $(\widehat{L}, D(\widehat{L}))$ the extended generator of the process (x_t).

Following [12], for $A \in \mathcal{B}(\overline{X})$ define processes p, p^* and $\widetilde{p}$ as follows:

$$p(t, A) = \sum_{k=1}^{\infty} I_{(t \geq T_k)} I_{(x_{T_k} \in A)};$$
$$p^*(t) = \sum_{k=1}^{\infty} I_{(t \geq T_k)} I_{(x_{T_k^-} \in \partial X)};$$
$$\widetilde{p}(t, A) = \int_0^t R(x_s, A)\lambda(x_s)ds + \int_0^t R(A, x_{s-})dp^*(s) = \sum_{T_k \leq t} R(x_{T_k -}, A).$$

Note that p, p^* are counting processes, $p^*(t)$ counting the number of jumps from the boundary of the process (x_t). $\widetilde{p}(t, A)$ is the compensator of $p(t, A)$ (see [12] for more explanations). The process $q(t, A) = p(t, A) - \widetilde{p}(t, A)$ is a local martingale.

Theorem 11 *Let H be an GSHM as in definition 7. Then the domain $D(L)$ of the extended generator L of H, as a Markov process, consists of those measurable functions f on $X \cup \partial X$ satisfying:*

1. *$f : \overline{X} \to \mathbb{R}$, $\mathcal{B}-$measurable; $t \to f(x_t^i(\omega_i))$ should have second order derivatives on $[0, S^i(\omega_i))$, for all $\omega_i \in \Omega^i$;*
2. *Boundary condition*

$$f(x) = \int_{\mathbb{X}} f(y)R(x, dy)), \ x \in \partial X;$$

3. *$Bf \in L_1^{loc}(p)$, where*

$$Bf(x, s, \omega) := f(x) - f(x_{s-}(\omega));$$

For $f \in D(L)$, Lf is given by

$$Lf \ (x) = L_{cont} f(x) + \lambda(x) \int_{\overline{X}} (f(y) - f(x))R(x, dy) \tag{3.8}$$

where

$$L_{cont} \ f(x) = \mathcal{L}_b f(x) + \frac{1}{2} Tr(\sigma(x)\sigma(x)^T \mathbb{H}^f(x)).$$

Proof. Let $(\widetilde{L}, D(\widetilde{L}))$ denote the extended generator of (x_t). We want to show that $(\widetilde{L}, D(\widetilde{L})) = (L, D(L))$. Suppose first that f satisfies 1-3. Then $Bf \in L_1^{loc}(\widetilde{p})$ and

$$\int_{[0,t]\times\overline{X}} Bf d\widetilde{p} = \int_{[0,t]} \int_{\overline{X}} (f(y)-f(x_s))R(x_s,dy)\lambda(x_s)ds + \\ + \int_{[0,t]} \int_{\overline{X}} (f(y)-f(x_{s-}))R(x_{s-},dy)dp^*(s).$$

Now the support of p^* is contained in the countable set $\{s : x_{s-} \in \partial X\}$ and because of the boundary condition 2. the second integral vanishes. Thus

$$\int_{[0,t]\times\overline{X}} Bf dq = \sum_{T_k\le t} (f(x_{T_k}) - f(x_{T_k-})) - \int_{[0,t]} \int_{\overline{X}} (f(y)-f(x_s))R(x_s,dy)\lambda(x_s)ds.$$

This process is a local martingale because of condition 3. Let T_m denote the last jump time prior or equal to t. Then

$$\sum_{T_k\le t} (f(x_{T_k}) - f(x_{T_k-})) = \{f(x_t) - f(x_{Tm}) + \sum_{k=1}^{m} (f(x_{T_k}) - f(x_{T_{k-1}}))\} \\ -\{f(x_t) - f(x_{Tm}) + \sum_{k=1}^{m} (f(x_{T_k-}) - f(x_{T_{k-1}}))\}.$$

The first bracketed term on the right is equal to $f(x_t) - f(x)$. To evaluate the second term, note that $x_{T_k-} = \phi(T_k - T_{k-1}, x_{T_{k-1}})$. Using Itô-formula, we get

$$f(x_{T_k-}) - f(x_{T_{k-1}}) = \int_{T_{k-1}}^{T_k} L_{cont} f(x_s)ds + \int_{T_{k-1}}^{T_k} < \sigma(x_s), \nabla f(x_s) > dW(s).$$

The second term is therefore equal to $\int_0^t L_{cont} f(x_s)ds + \int_0^t < \sigma(x_s), \nabla f(x_s) > dW(s)$ and we obtain

$$C_t^f := f(x_t) - f(x) - \int_0^t Lf(x_s)ds = \int_0^t < \sigma(x_s), \nabla f(x_s) > dW(s) + \int_{[0,t]\times\overline{X}} Bf dq$$

is a local martingale (the sum between a continuous martingale and a discrete martingale), where L is given by (3.8). Thus $f \in D(\widehat{L})$ and $\widehat{L}f = Lf$.

Conversely, suppose that $f \in D(\widehat{L})$. Then the process

$$M_t := f(x_t) - f(x) - \int_0^t h(x_s)ds$$

is a local martingale, where $h = \widehat{L}f$. It must be the case that M_t to be the sum between a continuous martingale M_t^c and a discrete martingale M_t^d. From [12] (Theorem (26.12), part 3, p.69), we have that $M_t^d = M_t^\rho$ for some predictable integrand $\rho \in L_1^{loc}(p)$, where M_t^ρ is given by

$$M_t^\rho = \int_{\overline{X}\times\mathbb{R}_+} \rho I_{(s\le t)} dq = \sum_{T_k\le t} \\ \rho(x_{T_k}, T_k, \omega) - \int_0^t \int_{\overline{X}} \rho(y,s,\omega)R(x_s,dy)\lambda(x_s)ds - \int_0^t \int_{\overline{X}} \rho(y,s,\omega)R(x_{s-},dy)dp^*(s).$$

Since M_t^d and M_t^ρ agree, their jumps ΔM_t^d and ΔM_t^ρ must agree; these only occur when $t = T_k$ for some k and are given by

$$\begin{aligned} \Delta M_t^d &= f(x_t) - f(x_{t-}) \\ \Delta M_t^\rho &= \rho(x_t,t,\omega) - \int_{\overline{X}} \rho(y,t,\omega)R(x_{t-},dy)I_{(x_{t-}\in\partial X)}. \end{aligned}$$

Thus $\rho(x_t,t,\omega) = f(x_t) - f(x_{t-})$ on the set $(x_{t-} \notin \partial X)$, which implies that $\rho(x,t,\omega) = f(x) - f(x_{t-})$ for all (x,t) except perhaps a set to which the process 'never jumps', i.e. $G \subset \mathbb{R}_+ \times X$ such that

$$E_z \int_G p(dt,dx) = 0, \ \forall z \in X.$$

Suppose that $z = x_{t-} \in \partial X$. Then equating ΔM_t^d and ΔM_t^ρ gives

$$f(x_t) - f(z) = \rho(x_t,t,\omega) - \int_{\overline{X}} \rho(y,t,\omega)R(z,dy)$$

and hence

$$f(x) - f(z) = \rho(x,t,\omega) - \int_{\overline{X}} \rho(y,t,\omega)R(z,dy)$$

except on a set $A \in \mathcal{B}(X)$ such that $R(z, A) = 0$. Integrating both sides of the previous equality with respect to $R(z, dx)$, we obtain (using the fact that $R(z, \cdot)$ is a probability measure)

$$\int_{\overline{X}} f(x)R(z, dx) - f(z) = \int_{\overline{X}} \rho(x, t, \omega)R(z, dx) - \int_{\overline{X}} \rho(y, t, \omega)R(z, dy) = 0.$$

Thus f satisfies the boundary condition. Now define (for fixed z)

$$\widetilde{\rho}(x, t, \omega) = \rho(x, t, \omega) - (f(x) - f(z)).$$

Using the boundary condition we get $\int_{\overline{X}} \widetilde{\rho}(y, t, \omega)R(z, dy) = \int_{\overline{X}} \rho(y, t, \omega)R(z, dy) = \widetilde{\rho}(x, t, \omega)$. Then

$$\widetilde{\rho}(x, t, \omega) = \int_{\overline{X}} \widetilde{\rho}(y, t, \omega)R(z, dy).$$

However, the right-hand side does not depend on x, and hence $\widetilde{\rho}(x, t, \omega) = u(t, \omega)$ for some predictable process u. The general expression for ρ is thus

$$\rho(x, t, \omega) = f(x) - f(x_{t-}) + u(t, \omega)I_{(x_{t-} \in \partial X)}.$$

Inserting this in the expression of M_t^ρ we find that M_t^ρ does not depend on u, therefore we can take $u \equiv 0$, obtaining $\rho = Bf$; hence the part 3 of theorem is satisfied.

Finally, consider the sample paths of M_t, $M_t^{Bf} + M_t^c$, for $t < T_1(\omega)$, starting at $x \in X$. We have

$$M_t = f(x_t(\omega_{i_0})) - f(x) + \int_0^t h(x_s(\omega_{i_0}))ds$$

while, because $p = p^* = 0$ on $[0, T_1)$,

$$M_t^{Bf} = -\int_{[0,t]} \int_{\overline{X}} (f(y) - f(x_s(\omega_{i_0})))R(x_s(\omega_{i_0}), dy)\lambda(x_s(\omega_{i_0}))ds.$$

So, since $M_t = M_t^{Bf} + M_t^c$ for all t a.s., it must be the case that $M_t = M_t^c$ for $t \in [0, T_1)$ and the generator coincides with the generator L_{cont} associated to the stochastic equation, the function $f(x_t(\omega_{i_0}))$ should have second order derivatives on $[0, T_1)$. The general case follows by concatenation. Calculations as before now show that for all $t \geq 0$,

$$M_t^{Bf} + M_t^c = f(x_t) - f(x) - \int_0^t Lf(x_s)ds$$

with L given by (3.8). Hence $f \in D(L)$ and $Lf = \widehat{L}f$. This completes the proof.

Chapter 4 Reachability Problem

4.1 Measurability of Reach Events

In a probabilistic framework, the reachability problem consists in determining the probability that the system trajectories enter some prespecified set starting from a certain set of initial conditions with a given probability distribution.

Let H be a GSHM as in the definition 7. Let E a Borel set of the state space X, i.e. $E \in \mathcal{B}(X)$. Let us to define reachable "event" associated to E:

$$\begin{aligned} Reach_T(E) &= \{\omega \in \Omega \mid \exists t \in [0,T] : x_t(\omega) \in E\} \\ Reach_\infty(E) &= \{\omega \in \Omega \mid \exists t \geq 0 : x_t(\omega) \in E\} \end{aligned}$$

The aim of this section is to show that $R_T(E)$ and $R_\infty(E)$ are measurable sets in the underlying probability space. Thus the computation of the probabilities $P[R_T(E)]$ and $P[R_\infty(E)]$ makes sense..

Proposition 12 *Let $E \in \mathcal{B}(X)$ be a given Borel set. Then $Reach_T(E)$ and $Reach_\infty(E)$ are universally measurable sets in Ω.*

Proof. Since any GSHM is strong Markov process with càdlàg property, the proposition is a direct corollary of the theorem 4.1 from [9].

4.2 Computation of Reach Event Probabilities

The basic idea of the reachability method proposed here is to employ the correspondence between some 'nice' Markov processes (like our process) and some quadratic forms, called *Dirichlet forms*, defined using the process generator. A Dirichlet form comes with the so-called notion of *capacity*, which is, roughly speaking, a nonlinear extension of a measure. The capacity associated with a Dirichlet form is in a very close connection with the hitting times of the corresponding Markov process. We investigate the possible benefits of applying a Dirichlet form based method to study the reachability problem of GSHM.

4.2.1 Target Sets

Usually a target set E in the state space is a level set for a given function $F : X \to \mathbb{R}$, i.e.

$$E = \{x \in X | F(x) > l\};$$

(F can be chosen as the Euclidean norm or as the distance to the boundary of E). The probability of the set of trajectories which hit E until time horizon $T > 0$ can be expressed as

$$P\{\sup_{t\in[0,T]} F(x_t) > l\}. \tag{4.1}$$

4.2.2 Induced Dirichlet Forms

The strong Markov property of GSHM, proved in the proposition 4.2.2 allows us to point out a quadratic form associated with it, as follows.

Let H be a GSHM as in the definition 7. Let $D(L)$ be the domain of its generator L (given by 3.8).

Using the Lebesgue measure λ^i on $\mathbb{R}^{d(i)}$, we define a measure m on $\mathcal{B}(X)$ such that for each $i \in Q$ the projection of m to X^i is exactly $\lambda|_{X^i}$, i.e.

$$m(\{i\} \times A) := \lambda^i(A),\ Q' \subset Q,\ A \in \mathcal{B}(\mathbb{R}^{d(i)}).$$

Let m^* the image of m through the map F.

Remark 7 *The process semigroup (P_t) may be viewed as a strongly continuous semigroup of operators on $L^2(X,m)$. Its generator is defined by the same limit (3.6) with respect to the norm of $L^2(X,m)$. The domain of the generator consists of those $f \in L^2(X,m)$ for which the limit (3.6) exists in the strong sense.*

Remark 8 *[18] Under the standard assumptions of section 3.2, there exists a* quasi-regular Dirichlet form[1] $(\mathcal{E}, D[\mathcal{E}])$ *on $L^2(X,m)$ associated with the process (x_t), given by*

[1] See the definition 3.1 from [18]

$$\begin{cases} D(L) \subset D[\mathcal{E}] \\ \mathcal{E}(u,v) = (-Lu,v),\ u \in D(L),\ v \in D[\mathcal{E}]. \end{cases}$$

We can think of a Dirichlet form $\mathcal{E}$ as a recipe for a Markov process $(\alpha_t)_{t\geq 0}$, in the sense that $\mathcal{E}$ describes the behavior of the composed process $u(\alpha_t)$ for every u in the domain of $\mathcal{E}$. There is no guarantee that the 'coordinates' $(u(\alpha_t))_u$ can be put together in a consistent way to form a process with reasonable sample paths.

Let us denote the sub-σ-algebra of $\mathcal{B}$, generated by F by $\sigma(F)$, and the projection operator from $L^2(X,\mathcal{B},m)$ to $L^2(X,\sigma(F),m)$ by $\mathcal{F}$; $\mathcal{F} \equiv E_m[\cdot|F]$ in case m is a probability measure.

An upper estimation for the probability (4.1) will be given in terms of the Dirichlet form induced by F on $\mathbb{R}$. This form corresponds to the process $F(x_t)$.
More precisely, the function F induces a form $\mathcal{E}^*$ on $L^2(\mathbb{R},m^*)$ by

$$\mathcal{E}^*(u^*,v^*) = \mathcal{E}(u^* \circ F, v^* \circ F);\ u^*,v^* \in D[\mathcal{E}^*]$$

where

$$D[\mathcal{E}^*] = \{u^* \in L^2(\mathbb{R},m^*) | u^* \circ F \in D[\mathcal{E}]\}.$$

Proposition 13 *[17] If $\mathcal{F}(\mathcal{D}) \subset D[\mathcal{E}]$ where $\mathcal{D}$ is some L^2-dense subset of $D[\mathcal{E}]$, then $\mathcal{E}^*$ is a Dirichlet form on $L^2(\mathbb{R},m^*)$.*

Assumption 5 *Suppose that the Dirichlet form $\mathcal{E}^*$ is quasi-regular[2].*

In [17], it is shown that, under a mild condition on the function F, the assumption 5 can be accomplished. This assumption ensures that there exists a right Markov process, (x_t^*), with the state space $\mathbb{R}$, associated with the Dirichlet form $\mathcal{E}^*$ [1]. If $F(x_t)$ happens to be Markovian then $\mathcal{E}^*$ is its associated Dirichlet form (see [23], for conditions on F which imply the Markov property of $F(x_t)$).

Assumption 6 *Suppose that the Dirichlet forms $\mathcal{E}$, $\mathcal{E}^*$ are symmetric[3]*

$$\begin{aligned} \mathcal{E}(u,v) &= \mathcal{E}(v,u),\ u,v \in D[\mathcal{E}]; \\ \mathcal{E}^*(u^*,v^*) &= \mathcal{E}^*(v^*,u^*),\ u^*,v^* \in D[\mathcal{E}^*]. \end{aligned}$$

Assumption 6 is not restrictive (any result valid for regular Dirichlet forms and invariant under quasi-homeomorphisms is applicable to quasi-regular Dirichlet forms [11]).

Each (quasi-regular) symmetric Dirichlet form can be expressed as the sum of its parts: continuous, jumping and killing corresponding to the same parts of the Markov process considered. Precisely, a regular Dirichlet form $\mathcal{E}$ can be decomposed using *the Beurling-Deny representation* [14]:

$$\begin{aligned} \mathcal{E}(u,v) &= \mathcal{E}_c(u,v) + \int_{X\times X\setminus d} [u(x)-u(y)][v(x)-v(y)]J(dx,dy) + \qquad (4.2) \\ + \int_X u(x)v(x)k(dx),\ u,v &\in D[\mathcal{E}] \cap C_0(X). \end{aligned}$$

Here $\mathcal{E}_c$ is a symmetric form with domain $D[\mathcal{E}_c] = D[\mathcal{E}]$ which satisfies the property
$\mathcal{E}_c(u,v) = 0$ if $u,v \in D[\mathcal{E}]$ have support compact and v is constant on a neighbourhood of $supp[u]$
J is a symmetric positive measure on $X \times X\setminus d$, d being the diagonal; and k is a positive measure on X. The form $\mathcal{E}_c and$ measures J and k are uniquely determined by $\mathcal{E}$; $\mathcal{E}_c$ is called the diffusion part of $\mathcal{E}$, and J and k are called the *jump* measure and the *killing* measure, respectively, associated with $\mathcal{E}$.
Note if E^* is open in $\mathbb{R}$ and $E = F^{-1}(E^*)$ then we can define for $p > 0$, the *p-capacity* of E

$$Cap_p(E) = \inf\{\mathcal{E}(u,u) + p(u,u)_m | u \in D[\mathcal{E}],\ u \geq 1\ m - a.e. \text{ on } E\} \qquad (4.3)$$

where $(\cdot,\cdot)_m$ is the inner product of $L^2(X,m)$ and the *p-capacity* of E^*

$$Cap_p^*(E^*) = \inf\{\mathcal{E}^*(u^*,u^*) + p(u^*,u^*)_{m^*} | u^* \in D[\mathcal{E}^*],\ u^* \geq 1\ m^* - a.e. \text{ on } E^*\} \qquad (4.4)$$

where $(\cdot,\cdot)_{m^*}$ is the inner product of $L^2(\mathbb{R},m^*)$.

Proposition 14 *[17] Under the assumptions 5 and 6, if E^* is open and $E = F^{-1}(E^*)$ then*

$$Cap_p(E) \leq Cap_p^*(E^*). \qquad (4.5)$$

We can consider the two first hitting times T_E (with respect to (x_t)) and T_{E^*} (with respect to (x_t^*)). Intuitively, the capacity (4.3) (resp. (4.4)) is the Laplace transform of the hitting time T_E (resp. T_{E^*}) of the target set (resp. of the 'induced' target set).

[2] See the definition 3.1 from [18]
[3] See [14, 18] for the theory of symmetric and non-symmetric Dirichlet forms.

4.2.3 Upper Bounds for Reach Set Probabilities

Assume that $m(X) < \infty$, $1 \in D[\mathcal{E}]$ and $k(X) < \infty$, where the killing measure k is described in the Beurling-Deny representation (4.2).

The translation of the capacitary inequality (4.5) into probabilistic terms for the right Markov processes (x_t) and (x_t^*) associated with $\mathcal{E}$ and $\mathcal{E}^*$ gives rise to the following result:

Proposition 15 *[17] Under the assumption 7, if $E^* \subset \mathbb{R}$ is an open set of finite Cap^*-capacity and $E = F^{-1}(E)$ then for all $p > 0$,*

$$P_m(T_E \leq T) \leq e^p\{E_{m^*}e^{-pT_{E^*}/T} + Tp^{-1}\int_{\mathbb{R}} E_r e^{-pT_{E^*}/T}k^*(dr)\}.$$

where k^* is the killing measure associated with the killing part of $\mathcal{E}^*$. Also

$$P_m(T_E \leq T) \leq p^{-1}e^p \min\{T\mathcal{E}^*(u^*, u^*) + p(u^*, u^*)_{m^*} | u^* \in D[\mathcal{E}^*],\ u^* \geq 1,\ m^* - a.e.\ on\ E^*\}$$

One might, for instance, use the small induced processes rather then the huge original process to deal with the reachability problem. The induced Dirichlet form capacity (of $E^* = (l, \infty)$) plays an essential role in obtaining the reach event probability estimation. If the model H is discretized then the induced process is a one-dimensional jump process and therefore the computation of Laplace transform and the mean level-crossing time is feasible. It is interesting to note that the capacity of the target set is subadditive. So even if the target set were very complex, then the capacity of target set is at most the sum of capacities of its parts.

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